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On the evaluation of barotropic–baroclinic instability parameters of zonal flows on a beta-plane

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The paper is concerned with the problem of the linear stability of an arbitrary inviscid zonal flow on a β-plane. Based on the analysis of integral relations following from the linear boundary-value problem, new evaluations, considerably more exact than the previously known ones, of the parameter region of unstable disturbances are derived. Some new relations among these bounds are established.

1. Introduction

The barotropic–baroclinic instability of large-scale zonal flows is the classical problem of geophysical hydrodynamics. The problem of examining the perturbation evolution of a certain prescribed, initial state and finding the bounds on the parameters of perturbations growing with time, is hardly tractable analytically for real flows with an arbitrary dependence on the meridional and vertical coordinates even within the linear theory. Until now only an asymptotic analysis of some particular situations has been carried out by Killworth (1980). This approach yields a physical understanding of the main mechanisms of instability, but gives no hope of describing linear instability over a wide range of basic parameters. A unique strongly idealized model proposed by Charney (1947) only allows us to carry out a sufficiently complete analysis of the linear boundary – value problem for the case of baroclinic instability. Moreover, even direct numerical analysis of the boundary-value problem in the general barotropic–baroclinic case is rather difficult; only the simplest two-/three-layer models have as yet been studied (Killworth 1980). Thus, the possibility of formulating *a priori* evaluations (instead of direct numerical treatment of the boundary-value problem, or in combination with it at a 'precomputing' stage) is of special interest.

The approach based on the analysis of integral relations for normal modes, which was initiated by Rayleigh (1880), has a long and rich tradition in hydrodynamics. But in the study of flow stability on a beta-plane the main achievements in this direction can be easily listed. They are connected with two works: Pedlosky (1964) and Miles (1964). Pedlosky (1964) derived an extension of Howard's (1961) semicircle theorem (proved for parallel stratified unrotating shear flows) for unstable modes of the barotropic–baroclinic instability problem. Pedlosky's semicircle theorem gives a bound on the phase velocity c of unstable disturbances in terms of the extreme values of the flow velocity. Pedlosky (1964) also constructed an independent growth-rate bound in terms of the maximum flow gradients. In work concerned with the problem of baroclinic instability, within Charney's (1947) model, Miles (1964) proved another semicircle theorem (here and henceforth we mean Theorem III of his work), also in terms of the extreme velocity values.

In our work, which follows a similar path, new bounds on the phase speed and the growth rate of unstable disturbances, as well as their interrelations, are derived. Some elements of a technique similar to that applied in recent works on the stability of stratified (Taylor–Goldstein) shear flows (Kochar & Jain 1979; Makov & Stepanyants 1984) have also been used.

The paper is organized as follows: first, in §2 we state the problem, review the main points of the key works (Pedlosky 1964; Miles 1964), and improve Pedlosky's semicircle theorem by extending Miles' theorem for the general case considered here (i.e. for barotropic–baroclinic instability of an arbitrary zonal flow). In §3 we prove a new inequality relating two functionals, which allows us to construct some effective bounds on the phase speed. Their general feature is that the domain of allowed speed sharply diminishes as β increases and tends to zero as non-dimensional β tends to infinity. Then in §4 some refinements of Miles' semicircle theorem based on the simultaneous use of Miles' results, our results of the previous section, and Pedlosky's semicircle theorem are derived. Section 5 deals with the relations between the different kinds of bounds. New relations are established as well as new bounds that result from these relations. In the conclusion, the main points are summarized and discussed.

2. The statement of the problem. Extension of Miles' theorem

2.1. *Statement of the problem*

Let us consider the problem of linear stability of inviscid baroclinic zonal flows in a channel on a β-plane. We shall follow exactly the original formulation of the problem by Pedlosky (1964), and preserve all the notation (Pedlosky 1979), where the underlying physical concepts for this class of problems are thoroughly discussed. We study propagation of wave-like disturbances of the form

$$\phi(y,z,x,t) = \phi(y,z)\exp[ik(x-ct)],$$

where ϕ denotes any disturbance field characteristics, k is the zonal wavenumber, and $c(c = c_r + ic_i)$ is the phase speed of the corresponding Fourier component. Axes x, y, z are oriented as follows: x is directed eastward, y northward, z vertically upward. The boundary-value problem in terms of $N(y,z,k,c)$ (the amplitude of northward displacement) with standard inviscid boundary conditions on the rigid lower boundary ($z = \eta(y)$), on the channel sidewalls ($y = \pm 1$), and on the upper free surface ($z = z_T$) has the well-known form

$$\partial_y[(U-c)^2\partial_y N] + \frac{1}{\rho_S}\partial_z[(U-c)^2\frac{\rho_S}{S}\partial_z N] - k^2N(U-c)^2 + \beta N(U-c) = 0, \quad (2.1)$$

$$N = 0, \quad y = \pm 1, \quad (2.2)$$

$$(U-c)^2\partial_z N + (U-c)SN\partial_y\eta = 0, \quad z = 0, \quad (2.3)$$

$$\partial_z N = 0, \quad z = z_T. \quad (2.4)$$

All the variables are non-dimensionalized by introducing characteristic scales of flow velocity (U^0), horizontal (L) and vertical (H) flow variability. The non-dimensional stratification parameter S is composed of a Brunt–Väisälä frequency $\varkappa$ and the Coriolis parameter f_0 at a certain fixed latitude y_0

$$S = \varkappa^2H^2/f_0^2L^2,$$

where
$$N^2 = -\frac{g}{\rho_S}\frac{d\rho_S}{dz}, \quad f_0 = f|_{y=y_0}$$

and ρ_S is a 'standard' density vertical distribution. Parameter β is the non-dimensional northward gradient of the Coriolis parameter at the same latitude y_0

$$\beta = \beta_0 L^2/U^0, \quad \text{where} \quad \beta_0 = \left.\frac{\partial f}{\partial y}\right|_{y=y_0}.$$

The problem is to get *a priori* constraints on the complex phase speed c for unstable perturbations.

2.2. *The work of Pedlosky* (1964)

Before proceeding with our analysis of the boundary-value problem (2.1)–(2–4), we should recall the main points of the work by Pedlosky (1964). For the unstable modes, i.e. with $c_i \neq 0$, there are two integral relations which are immediate consequences of (2.1)–(2.4):

$$\langle UP\rangle = c_r\langle P\rangle + \tfrac{1}{2}\beta\langle j\rangle + \int_{-1}^{1} j(y,0)\,\partial_y\eta\,dy, \tag{2.5}$$

$$\langle U^2P\rangle = (c_r^2+c_i^2)\langle P\rangle + \beta\langle Uj\rangle + \int_{-1}^{1} j(y,0)\,U\partial_y\eta\,dy, \tag{2.6}$$

where
$$\langle f\rangle = \int_{-1}^{1}\int_0^{z_T} f\,dy\,dz; \quad j = \rho_S|N|^2,$$
$$P = \rho_S(S^{-1}|\partial_z N|^2 + |\partial_y N|^2 + k^2|N|^2).$$

The integral relations (2.5), (2.6) are written here for the sake of generality to take account of bottom topography. We shall, however, confine ourselves to consideration of flows over a flat bottom (i.e. $\partial_y\eta = 0$). These relations (with the late terms omitted) form the basis for all the subsequent analysis in Pedlosky (1964), Miles (1964) and in our treatment of the problem.

In Pedlosky (1964) the key tool is the inequality

$$\langle P\rangle \geqslant (k^2+\tfrac{1}{4}\pi^2)\langle j\rangle, \tag{2.7}$$

which relates specifically averaged values P and j and gives an evaluation from below of $\langle P\rangle$ in terms of $\langle j\rangle$. Rewriting (2.5) in the form

$$c_r = \frac{\langle UP\rangle}{\langle P\rangle} - \tfrac{1}{2}\beta\frac{\langle j\rangle}{\langle P\rangle} \tag{2.8}$$

and applying the inequality (2.7), one can easily find that

$$U_{min} - \frac{\beta}{2(k^2+\frac{1}{4}\pi^2)} \leqslant c_r \leqslant U_{max}, \tag{2.9}$$

where U_{max}, U_{min} are the maximum and the minimum values of U in the meridional plane.

Some manipulations with the evident inequality

$$0 \geqslant \langle(U-U_{max})(U-U_{min})P\rangle = \langle(U^2 - U(U_{max}+U_{min}) + U_{max}U_{min})P\rangle \tag{2.10}$$

taking into account (2.5), (2.6), yield

$$0 \geqslant [c_r^2+c_i^2-c_r(U_{max}+U_{min})+U_{max}U_{min}]\langle P\rangle + \beta\langle j[U-\tfrac{1}{2}(U_{max}+U_{min})]\rangle. \tag{2.11}$$

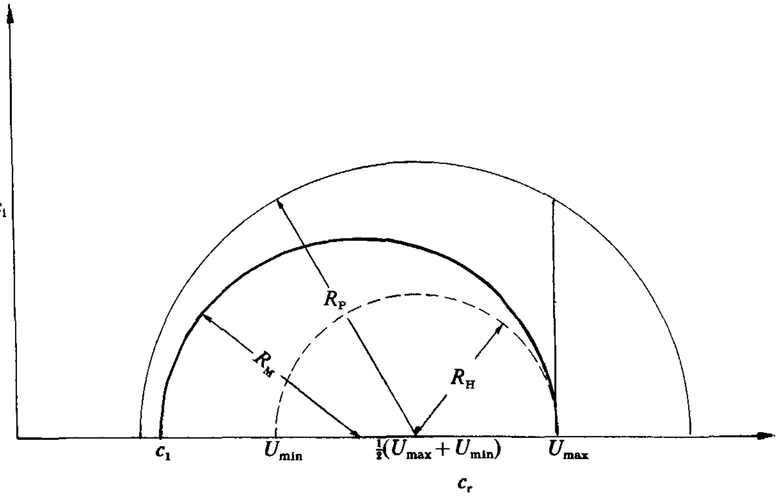

FIGURE 1. Miles' and Pedlosky's bounds on the complex phase speed c of unstable disturbances. The sketch illustrates the situation when $\beta > \beta_{MP}$ and Miles' semicircle (2.20) (the thick line) lies within Pedlosky's bounds (2.9), (2.12) (the thin lines). Howard's semicircle is shown for scale (the dashed line).

The last term in (2.11) is evaluated straightforwardly via replacement of U by U_{min}. Finally, on dividing (2.11) by the positive integral $\langle P \rangle$, we get from (2.11)

$$\left.\begin{aligned} [c_r-\tfrac{1}{2}(U_{max}+U_{min})]^2+c_i^2 &\leqslant R_P^2, \\ R_P^2 = [\tfrac{1}{2}(U_{max}-U_{min})]^2 &+ \frac{\beta}{(k^2+\frac{1}{4}\pi^2)}\tfrac{1}{2}(U_{max}-U_{min}). \end{aligned}\right\} \tag{2.12}$$

Thus, the complex phase speed c of unstable disturbances lies within a semicircle in the c-plane, with radius R_P and with its centre on the real axis at the mean velocity (see figure 1). The portion of the semicircle for which $c_r > U_{max}$ is prohibited for the eigenvalues c by the inequality (2.9). We note that the bounds on c, given by the inequalities (2.9), (2.12) depend only on the two basic state parameters, namely on the maximum and the minimum values of the flow velocity attained by U in the meridional plane. These bounds also depend implicitly on the flow location (R_P depends on β and therefore on latitude); it also should be pointed out that the semicircle expands up to infinity when β increases.

An alternative evaluation in terms of other flow characteristics has been derived by Pedlosky (1964) from the boundary-value problem (2.1)–(2.4), rewritten in terms of a new variable χ

$$\chi(y,z) = N(y,z)(U(y,z)-c)^{\frac{1}{2}}. \tag{2.13}$$

From this form of the boundary-value problem there immediately follows an integral relation

$$\langle \rho_S(S^{-1}|\chi_z|^2+|\chi_y|^2+k^2|\chi|^2)\rangle = \left\langle [S^{-1}(U_z)^2+(U_y)^2]\frac{\rho_S|\chi|^2}{4|U-c|_2}\right\rangle, \tag{2.14}$$

which, in combination with an inequality similar to (2.7), yields the upper bound denoted as C_P for the disturbance growth rate in terms of a certain combination (denoted as q) of flow gradients

$$c_i^2 \leqslant \tfrac{1}{4}q(k^2+\tfrac{1}{4}\pi^2) \equiv C_P; \quad q \equiv [S^{-1}(U_z)^2+(U_y)^2]_{max} \tag{2.15}$$

where the subscript max denotes the maximum value of the terms in the square bracket attained over the meridional plane.

2.3. *The extension of Miles* (1964)

Another semicircle theorem was proved by Miles (1964) (recall that we mean only theorem III of his work) in a paper concerned with the problem of zonal flow baroclinic instability within Charney's (1947) model. It was shown that, within this model Miles' theorem is stronger than that of Pedlosky.

We now extend this theorem to the general case and investigate its relation with Pedlosky's theorem, which is not as simple in the case of baroclinic instability. This extension can be made in a very straightforward manner; it is enough to point out that both Miles' theorem itself, and its proof hold true for the general problem of barotropic–baroclinic instability of an arbitrary zonal flow as well. For the sake of the further analysis, however, it is convenient to present the proof in detail.

Miles starts with the most general integral relation of a type similar to (2.10):

$$\langle P\rangle\{[c_r-\tfrac{1}{2}(c_1+c_2)]^2+c_i^2-\tfrac{1}{4}(c_2-c_1)^2\} = \langle P(U-c_1)(U-c_2)\rangle+\beta\langle j[\tfrac{1}{2}(c_1+c_2)-U]\rangle, \quad (2.16)$$

where c_1, c_2 are arbitrary real parameters and $c_2 > c_1$. The main idea is to select certain c_1, c_2 such that the right-hand side of (2.16) is negative, then the phase speed lies within a semicircle (for $c_i > 0$) based on the real axis diameter (c_1, c_2). On representing the right-hand side of (2.16) in the form

$$\langle Q(U-c_1)(U-c_2)\rangle+k^2\langle j(U-c_+)(U-c_-)\rangle, \quad (2.17)$$

where

$$c_\pm = \tfrac{1}{2}(c_1+c_2)+\tfrac{1}{2}\{[\tfrac{1}{2}(c_2-c_1)]^2+U_\beta^2\}^{\frac{1}{2}};$$

$$U_\beta = \beta/k^2; \quad Q = \rho_S(S^{-1}|N_z|^2+|N_y|^2), \quad (2.18)$$

Miles (1964) showed that the choice of constants

$$\left.\begin{aligned} c_1 &= U_{\min}-\tfrac{1}{2}U_\beta+(\tfrac{1}{2}U_\beta)^2(U_{\max}-U_{\min}+\tfrac{1}{2}U_\beta)^{-1}, \\ c_2 &= U_{\max} \end{aligned}\right\} \quad (2.19)$$

provides the negativity of the right-hand side of (2.16) and thus yields the family of semicircles

$$(c_r-\tfrac{1}{2}(c_1\text{-}c_2))^2+c_i^2 \leqslant [\tfrac{1}{2}(c_2-c_1)]^2, \quad (2.20)$$

with the minimum value of the semicircle radius R_M given as

$$R_M = \tfrac{1}{2}(c_2-c_1) = \tfrac{1}{2}[U_{\max}-U_{\min}+\tfrac{1}{2}U_\beta-(\tfrac{1}{2}U_\beta)^2(U_{\max}-U_{\min}+\tfrac{1}{2}U_\beta)^{-1}]. \quad (2.21)$$

It should be noted that R_M increases monotonically with β from Howard's radius value equal to $\frac{1}{2}(U_{\max}-U_{\min})$ when $\beta\to 0$, to twice this value as $\beta\to\infty$. Miles noted that these semicircles always lie within those of Pedlosky (figure 1), but this holds true only for the case of baroclinic instability. In the general situation an intersection of Miles', (2.20), and Pedlosky's, (2.12), semicircles can take place. Let us analyse the possible cases. It is easy to see that for the baroclinic instability (we omit the $\frac{1}{4}\pi^2$ in (2.19)) or for β large enough, Miles' semicircle lies within that of Pedlosky. When β equals zero both semicircles degenerate into Howard's semicircle. Intersection of the contours occurs at small β. The boundary value β_{MP} separates these cases. It corresponds to the two semicircles touching each other at the point $(c_1, 0)$ and can be found from the fourth-order equation

$$[R_H^2+R_H\beta/(k^2+\tfrac{1}{4}\pi^2)]^{\frac{1}{2}}-R_H = \tfrac{1}{2}U_\beta-(\tfrac{1}{2}U_\beta)^2(U_{\max}-U_{\min}+\tfrac{1}{2}U_\beta)^{-1}, \quad R_H = \tfrac{1}{2}(U_{\max}-U_{\min}).$$

It is easy to see that β_{MP} increases monotonically as k decreases, from zero at $k \to \infty$ to the limiting value β^0_{MP} at $k = 0$. This limit is given by the simple explicit expression

$$\beta^0_{\mathrm{MP}} = 2\pi^2 R_{\mathrm{H}} = \pi^2 (U_{\max} - U_{\min}). \tag{2.22}$$

Thus for $\beta > \beta^0_{\mathrm{MP}}$ all the semicircles (2.20) (i.e. for all k) lie within those of Pedlosky (2.12), while for $\beta < \beta^0_{\mathrm{MP}}$ an intersection of the contours (2.12), (2.20) necessarily takes place for some k, and the mutual region is the domain in question.

2.4. *The work of Hall* (1980)

To complete the picture of interrelations between the previously existing semicircle theorems, let us also mention the semicircle theorem proved by Hall (1980) for the barotropic instability problem of rectilinear barotropic flow upon a meridionally non-uniform bottom. This theorem can be easily extended to our problem, as was done with Miles' theorem. Within the context of our problem, Hall's semicircle theorem states that the domain of allowed c confined by his one-parametric family of semicircles is equivalent to the intersection of Pedlosky's (1964) semicircle and Miles' (1964) semicircle of theorem II (not theorem III). As theorem II yields the semicircle corresponding to the largest possible one of the family (2.20) (i.e. with radius equal to $2R_{\mathrm{H}}$ and with the same centre position) it is always weaker than theorem III, which we have extended here, and thus Hall's theorem imposes no additional constraints, compared to those of the previous subsection.

2.5. *Discussion*

Our goal is to get sharper constraints on c than the set (2.9), (2.12), (2.15), (2.20) and to establish relations between different types of constraints. The most evident weak point of Miles' and Pedlosky's evaluations (2.12), (2.20) seems to be the fact that both domains expand when β increases. This feature contradicts the generally accepted understanding of the underlying physics of instability (for a brief discussion of this issue, see §6.2); one of our particular aims is to construct evaluations free of this defect. We also note that in our study some additional flow characteristics will be involved in the analysis.

3. 'Quasi-parabola' bound and its implications

In this section we shall derive new bounds on the phase speed of unstable modes c by establishing new relations between the integrals $\langle P\rangle$ and $\langle j\rangle$ of (2.5), (2.6). Firstly, we formulate these relations in the form of the following lemma.

3.1. *The evaluation of* $\langle P\rangle$ *in terms of* $\langle j\rangle$ *from above*

LEMMA. *For unstable modes the following inequality holds*:

$$\langle P\rangle \leqslant \frac{G}{c_i^2}\langle j\rangle, \tag{3.1}$$

where

$$G = \tfrac{1}{2}\{q + l(q - 4k^2 c_i^2)^{\frac{1}{2}}\}; \quad l = |U_y|_{\max} + |S^{-\frac{1}{2}} U_z|_{\max}. \tag{3.2}$$

Proof. Let us differentiate the function χ, which is defined by the relation (2.13), with respect to y:

$$\chi_y = N_y (U-c)^{\frac{1}{2}} - \tfrac{1}{2} N U_y (U-c)^{-\frac{1}{2}}, \tag{3.3}$$

then the following inequality can be written:

$$|\chi_y|^2 \geqslant |U-c|\,|N_y|^2 + \tfrac{1}{4}(U_y)^2|N|^2|U-c|^{-1} - |U_y|\,|N|\,|N_y|. \tag{3.4}$$

Similarly, differentiation with respect to z yields

$$|\chi_z|^2 \geqslant |U-c|\,|N_z|^2 + \tfrac{1}{4}(U_y)^2|N|^2|U-c|^{-1} - |U_y|\,|N|\,|N_y|. \tag{3.5}$$

Substituting (3.4), (3.5) into (2.14) we obtain

$$B^2_{(y)} + B^2_{(z)} \geqslant B^2_{(y)} + B^2_{(z)} + E^2_{(y)} + E^2_{(z)} + k^2 D^2 - \langle \rho_s |U_y|\,|N|\,|N_y|\rangle - \langle \rho_S S^{-1}|U_2|\,|N|\,|N_z|\rangle, \tag{3.6}$$

where

$$\left.\begin{aligned} B^2_{(y)} &= \left\langle \frac{(U_y)^2\rho_S|N|^2}{4|U-c|}\right\rangle; \quad B^2_{(z)} = \left\langle \frac{(U_z)^2\rho_S|N|^2}{4S|U-c|}\right\rangle;\\ E^2_{(y)} &= \langle \rho_S|U-c|\,|H_y|^2\rangle; \quad E^2_{(z)} = \left\langle \frac{\rho_S|U-c|\,|N_z|^2}{S}\right\rangle;\\ D^2 &= \langle \rho_S|U-c|\,|N|^2\rangle. \end{aligned}\right\} \tag{3.7}$$

Let us evaluate the two last integrals in (3.6), utilizing the Caushy–Bunyakovsky–Shwarts inequality twice:

$$\langle \rho_S|\,U_y|N|\,|N_y|\rangle \leqslant \left\langle \frac{\rho_S(U_y)^2|N|^2}{|U-c|}\right\rangle^{\frac{1}{2}} \langle \rho_S|N_y|^2\,|U-c|\rangle^{\frac{1}{2}}. \tag{3.8}$$

Similarly we get

$$\langle \rho_S S^{-1}|U_z|\,|N|\,|N_z z|\rangle \leqslant 2B_{(z)}E_{(z)}. \tag{3.9}$$

Let us strengthen the inequality (3.6) using (3.9), (3.8)

$$E^2_{(y)} + E^2_{(z)} + k^2D^2 - 2B_{(y)}E_{(y)} - 2B_{(z)}E_{(z)} \leqslant 0. \tag{3.10}$$

On rewriting (3.10) in a slightly different form,

$$(E_{(y)} - B_{(y)})^2 + (E_{(z)} - B_{(z)})^2 \leqslant B^2_{(y)} + B^2_{(z)} - k^2D^2, \tag{3.11}$$

we obtain

$$E_{(y)} \leqslant B_{(y)} + [B^2_{(y)} + B^2_{(z)} - k^2D^2]^{\frac{1}{2}}; \quad E_{(z)} \leqslant B_{(z)} + [B^2_{(y)} + B^2_{(z)} - k^2D^2]^{\frac{1}{2}}. \tag{3.12}$$

From (3.10), (3.12) we get

$$\begin{aligned} E^2_{(y)} + E^2_{(z)} + k^2D^2 &\leqslant 2B_{(y)}E_{(y)} + 2B_{(z)}E_{(z)}\\ &\leqslant 2\{B^2_{(y)} + B^2_{(z)} + (B_{(y)} + B_{(z)})[B^2_{(y)} + B^2_{(z)} - k^2D^2]^{\frac{1}{2}}\}. \end{aligned} \tag{3.13}$$

Finally, it follows from (3.13), that

$$\begin{aligned} c_i\langle P\rangle &= c_i\langle \rho_S(S^{-1}|N_z|^2 + |N_y|^2 + k^2|N|^2)\rangle\\ &\leqslant E^2_{(y)} + E^2_{(z)} + k^2D^2 \leqslant 2\{B^2_{(y)} + B^2_{(z)} + (B_{(y)} + B_{(z)})[B^2_{(y)} + B^2_{(z)} - k^2D^2]^{\frac{1}{2}}\}\\ &\leqslant \frac{1}{2c_i}\{[(U_y)^2 + S^{-1}(U_z)^2]_{\max} + |U_y|_{\max} + |S^{-\frac{1}{2}}U_z|_{\max}]\\ &\quad \times ([(U_y)^2 + S^{-1}(U_z)^2]_{\max} - 4k^2c_i^2)^{\frac{1}{2}}\}\langle j\rangle = \frac{G}{c_i}\langle j\rangle. \end{aligned} \tag{3.14}$$

Q.E.D.

We note that positivity of the expressions under the square roots in (3.2) or (3.13), (3.14) is provided by the inequality (2.15).

3.2. *Derivation of 'quasi-parabola' and 'composite quasi-parabola' bounds on c*

Let us reconsider the derivation of the evaluation (2.9). We recall that the upper bound on (2.9) was found by neglecting the definitely negative term proportional to β in (2.8) (the ratio $\langle j\rangle/\langle P\rangle$ is obviously positive). The lemma proved above allows us to get straightforwardly a lower bound for this ratio and thus to obtain a more precise upper bound for the complex phase speed c. From (2.8), (3.1) it follows that

$$c_r \leqslant U_{max} - \frac{\beta}{2G} c_i^2. \tag{3.15}$$

The curve that bounds the region defined by (3.15) starts at the point $(U_{max}, 0)$ and behaves as a quadratic parabola at small c_i, then c_i continues to increase monotonically up to the value $C_m (C_m = q^{\frac{1}{2}}/2k)$ with decreasing c_r. This maximum value of ordinate c_i is attained at a finite value of c_r, which we denote as $c_{rm} = U_{max} = U_{max} - \beta/4k^2$. For $c_r < c_{rm}$ the ordinate c_i preserves the same value C_m (see figure 2). We shall hereinafter refer to the curve segment between the points $(U_{max}, 0)$ and $(U_{max} - \frac{1}{4}U_\beta k^{-2}, C_m)$ as a quasi-parabola. We note that apart from being explicitly dependent on β and wavenumber k, this quasi-parabola segment depends on two different combinations of the extremal flow gradients q and l, while the abscissa of the conjunction point depends exclusively on U_β, and the ordinate of the half-line depends only on q and k, similarly to Pedlosky's growth-rate bound (2.15). It should be also pointed out, that the half-line of (3.15) is always somewhat above this bound. (It should be noted, however, that for an important particular class of problems, namely, for the problem of purely baroclinic instability, the growth-rate bound of the type (2.15) exactly coincides with the half-line of (3.15). Hence, *for the baroclinic case* the boundary curve (3.15) coincides with or lies below the baroclinic analogue of (2.15).) Thus, for the case considered here, the quasi-parabola segment of (3.15) always intersects Pedlosky's straight line at a certain point (c_{rP}, C_P). This fact allows us to construct a new bound by virtue of a combination of (3.15) and (2.15) as follows:

$$c_i^2 \leqslant \begin{cases} 2G(U_{max} - c_r), & c_{rP} \leqslant c_r \leqslant U_{max} \\ C_P^2, & c_r < c_{rP}, \end{cases} \tag{3.16}$$

where $\qquad c_{rP} = U_{max} - \beta/(4k^2+\pi^2)[1+\pi m(4k^2+\pi^2)^{-\frac{1}{2}}]; \quad m = lq^{-\frac{1}{2}}.$

For convenience we shall further refer to (3.15) and (3.16) as the 'quasi-parabola' bound and the 'composite quasi-parabola' bound, respectively. How meaningful these bounds are (in comparison with all other known bounds) and their implications will be considered below.

3.3. *The implications of the bounds* (3.15) *and* (3.16)

A bound should be considered meaningful when it diminishes the domain on the c-plane allowed by other bounds. It is also of special interest to get an *a priori* evaluation in terms of the maximum growth rate. Here we shall show that (3.15), (3.16) effectively cut off the instability domain prescribed by (2.15), (2.20) for a wide range of parameters. (Obviously the bound (3.15) always improves Pedlosky's result (2.12) and therefore is not discussed in this context here.) Combination of (3.15), (3.16) with Miles' semicircle theorem (2.20) will allow us to improve the growth-rate evaluation as well.

3.3.1. *A growth-rate evaluation*

It is easy to see that a combination of the quasi-parabola bounds (3.15), (3.16) and Miles' semicircle theorem allows us to find the intersection point of the boundary lines of (2.16), (3.15) and thus to obtain a growth-rate-type evaluation. This bound will clearly be more precise than (2.15), (2.16) (in growth-rate terms), when the abscissa of the intersection point exceeds c_{rP}. Then c_i^*, the ordinate in question, of the intersection point of the bound lines of (2.16), (3.15) can be found straightforwardly from the equation

$$(c_i^*)^2 - \frac{2GR_M}{\beta} + \frac{G^2}{\beta^2} = 0, \tag{3.17}$$

which is easily presented as a quadratic equation in terms of the new variable $(q-4k^2c_i^2)^{\frac{1}{2}}$. However, we shall not write down this bulky solution here; instead, we believe it more useful to give a qualitative analysis and to ascertain the relative positions of the quasi-parabola (3.15) and Miles' semicircle (2.20).

There are three qualitatively different relative positions of the curves in question, schematically depicted in figure 2.

The quasi-parabola touches Miles' semicircle from inside at the point $(U_{max}, 0)$ (see figure 2) when β is large enough, as prescribed by the inequality

$$\beta \geqslant \beta_1 = G_0 R_M^{-1}; \quad G_0 = G(c_i)|_{c_i=0}. \tag{3.18a}$$

Again, we would rather present a simple evaluation of the threshold value β_1, than a bulky solution of (3.18*a*). As R_M varies from Howard's value R_H to $2R_H$ the threshold value $\beta_1(k, U_{max}, U_{min}, q, l)$ can be easily evaluated as follows:

$$\frac{G_0}{U_{max}-U_{min}} \leqslant \beta_1 \leqslant \frac{2G_0}{U_{max}-U_{min}}. \tag{3.18b}$$

For a certain range of $\beta \in (\beta_2; \beta_1)$, the quasi-parabola touches the semicircle from outside at the point $(U_{max}, 0)$ and then crosses it at larger values of c_i (see figure 2*b*). The second threshold value $\beta_2(k, U_{max}, U_{min}, q, l)$ can be found from the equation

$$(\beta_2)^2 - \beta_2 4R_M k^2 + 4R_M^2 l^2 k^4/q - (q^2-l^2)k^2 = 0. \tag{3.19}$$

We note that when k is large enough, β_2 tends to zero.

For values of β smaller than β_2, the quasi-parabola goes above the semicircle and cannot be used for improvement of (2.20).

From the point of view of these bounds, refinement of another pair of threshold values of β (we refer to them as β_{P_1}, β_{P_2}) is important. They correspond to the situations of the quasi-parabola and the semicircle crossing, at abscissae equal to C_P, (see figure 2*d*, *e*). The values β_{P_1}, β_{P_2} can be derived from (3.17) by substituting C_P instead of c_i and then treating this equation as an equation for β:

$$\beta^2 C_P^2 - 2GR_M\beta + G^2 = 0. \tag{3.20}$$

The smaller root β_{P_1} gives the threshold of where the quasi-parabola bounds (3.15), (3.16) become meaningful in the range of β $(\beta_{P_1}, \beta_{P_2})$. The bounds (3.15), (3.16) cut off a certain part of the domain allowed by (2.15), (2.16), but give no refinement of the growth-rate bound. For the range of β exceeding the second threshold β_{P_2}, the

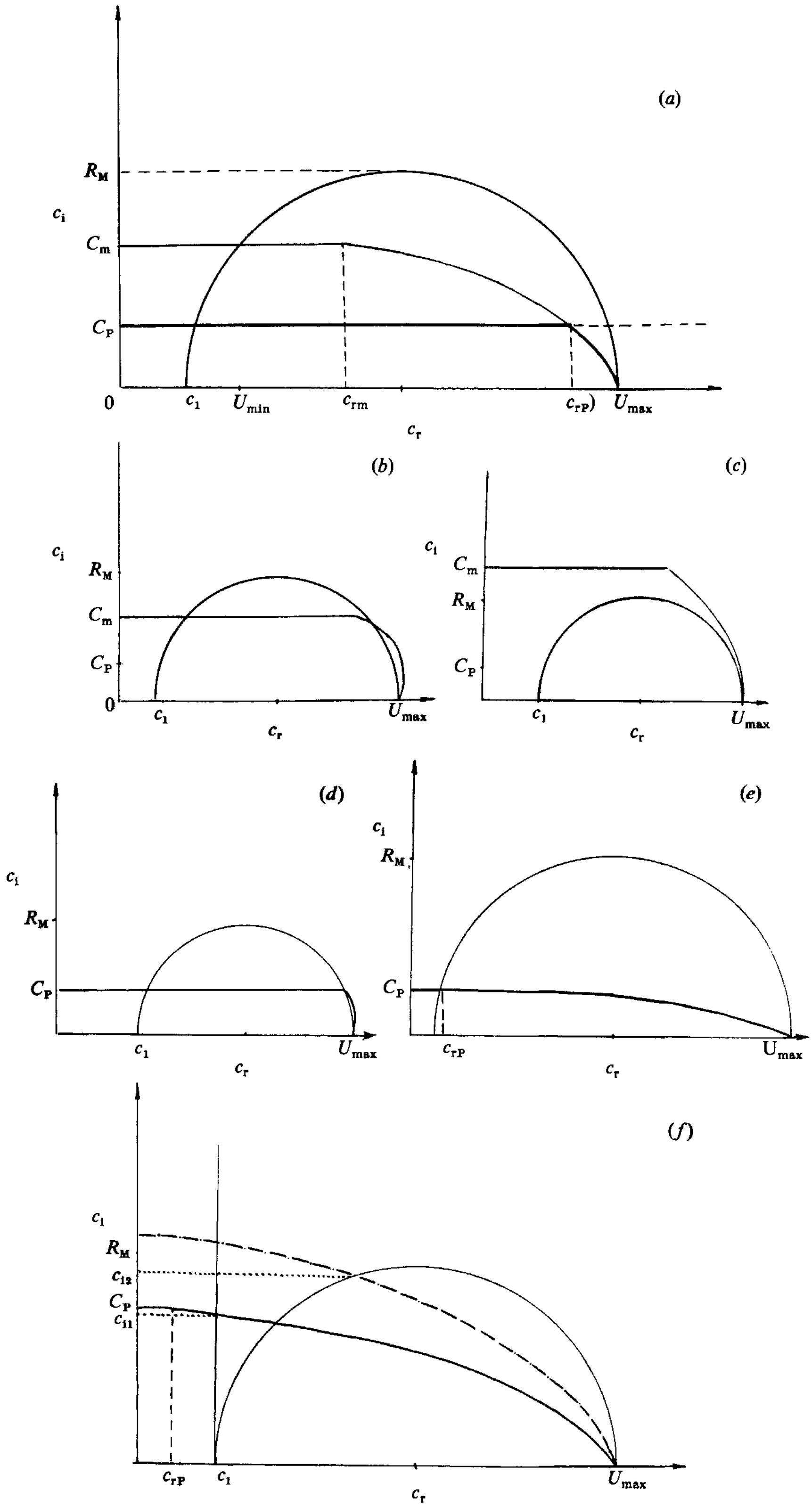

FIGURE 2. For caption see facing page.

quasi-parabola bounds allow us to get a more precise growth-rate bound as well. We note that the different threshold values of β are related as follows:

$$\beta_{P_2} > \beta_1 > \beta_{P_1} > \beta_2.$$

3.3.2. *Simplified versions of the growth-rate bound*

For many purposes, it is convenient to have a less precise but more simple explicit presentation of the growth-rate bound, than an exact solution to (3.17).

The most simple and therefore useful forms of the approximate solutions to (3.17) for the growth rate can be found through simple geometrical considerations. First, an upper bound for growth rate r can be obtained by calculating c_i at the intersection point (we shall refer to it as C_{i1}), of (3.15) and the vertical straight line touching Miles' semicircle from the side of smaller c_r values (figure 2). One obtains immediately from (3.15), (2.16) (see figure 2*e*)

$$(C_{i1})^2 \leqslant 4GR_M/\beta, \tag{3.21a}$$

or in terms of the growth rate

$$r^2 = (q + l(q-4r^2)^{\frac{1}{2}})X, \tag{3.21b}$$

where

$$X = 2R_M U_\beta^{-1}.$$

It should be stressed that the growth-rate bound given by (3.21*a*) depends on k and β only in combination: $U_\beta = \beta/k^2$. Let us normalize r ($\tilde{r}^2 = 4r^2/q$) and write down the explicit solution of the quadratic equation (3.21*b*):

$$\tilde{r}^2 = \tilde{X}(\tfrac{1}{2}(2-m\tilde{X}) + \{[\tfrac{1}{2}(2-m\tilde{X})]^2 + (m-1)]^{\frac{1}{2}}\}, \tag{3.22a}$$

where

$$\tilde{X} = 4X.$$

We point out a remarkable feature of (3.22): the upper bound for the growth rate r behaves oppositely to all known bounds, r decreases with β increasing (at least when β is large enough) and k fixed. It should be specially stressed that r decreases down to zero as β tends to infinity. We have for large β

$$\tilde{r}^2 = 2(1+m^{\frac{1}{2}})(U_{max} - U_{min})\,U_\beta^{-1} \tag{3.22b}$$

Obviously, the same asymptotics (3.22*b*) hold also for small k.

We note that for an important particular class of flows (namely, for purely barotropic or baroclinic ones), expression (3.22*a*) takes an especially simple form:

$$\tilde{r}^2 = \tilde{X}(2-\tilde{X}); \quad \tilde{X} \leqslant 1 \tag{3.22c}$$

Let us consider another possible way to construct a comparatively simple upper bound for the growth rate. We approximate the quasi-parabola bound (3.15) by a quadratic parabola (mentioned above as the small-c_i expansion). This parabolic approximation can also be of interest in itself:

$$c_i^2 \leqslant 2G_0(U_{max} - c_r)/\beta. \tag{3.23}$$

An evaluation of $(c_i)_{max}$, based on the calculation of the ordinate (C_{i2}) of the

FIGURE 2. The sketches (*a–e*) illustrate the positions of the 'quasi-parabola' bounds with respect to Miles' semicircle: —, the quasi-parabola bound (3.15) and the semicircle (2.20); —, the 'composite quasi-parabola' bound (3.16); ----, Pedlosky's growth-rate bound (2.15); –·–·–· parabolic approximation (3.23). (*a*) $\beta_1 < \beta$; (*b*) $\beta_2 < \beta < \beta_1$; (*c*) $\beta < \beta_2$; (*d*) $\beta = \beta_{P_1}$; (*e*) $\beta = \beta_{P_2}$; (*f*) An illustration to the parabolic approximation (3.23) and the evaluations (3.21*a*), (3.24).

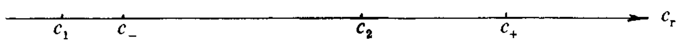

FIGURE 3. The relative positions of constants c_+, c_-, c_1, c_2 on the c_r-axis.

intersection point of the quadratic parabola and Miles' semicircle, appears to improve (3.22*a*) only in the large-β range. Therefore we present here the expression for $(c_i)_{max}$ valid only in this range.

$$(c_{i2}^*)^2 = \frac{2G_0 R_M}{\beta} - \frac{G_0^2}{\beta^2} \leqslant \frac{(q+lq^{\frac{1}{2}})(U_{max}-U_{min})}{\beta} - \frac{(q+lq^{\frac{1}{2}})^2}{4\beta^2}. \tag{3.24}$$

Similarly to (3.22*a*), the bound (3.24) increases down to zero with β increasing to infinity. But contrary to (3.22*a*), it exhibits no self-similar dependence on β and k.

4. Modifications of Miles' theorem

4.1. *Miles' formulation*

In this section we shall modify Miles' (1964) theorem, extended for our case in §2.3, exploiting his main idea and our results of the previous section, as well as the results of Pedlosky (1964), reviewed in §2.2.

We recall that Miles (1964) proved the phase speed c of the unstable disturbances to lie within a certain semicircle prescribed as follows:

$$[c_r - \tfrac{1}{2}(c_1+c_2)]^2 + c_i^2 \leqslant [\tfrac{1}{2}(c_2-c_1)]^2. \tag{4.1}$$

This statement holds for arbitrary c_1, c_2 satisfying conditions

$$c_1 \leqslant U_{min}; \quad c_- \leqslant U_{min}; \quad c_2 \geqslant U_{max}; \quad c_+ \geqslant U_{max}, \tag{4.2}$$

where c_+, c_- are specified in (2.17).

Miles' particular choice of c_1, c_2 (2.19), provides the minimum radius R_M of the semicircle (given by (2.22)), but, in view of combination of this constraint with the results of §3, it appears that another choice of these constants can lead to sharper constraints on c.

4.2. *Improvement of Miles' bound*: c_1 *is varied*, c_2 *is fixed*

We recall that while deriving (4.1), the term

$$\langle j(U-c_+)(U-c_-)\rangle$$

(which appears on the right-hand side of (2.16) presented in the form (2.17)) has been neglected as definitely non-positive (under the condition of the proper choice of c_+, c_- as functions of c_1, c_2; the positions of c_+, c_-, c_1, c_2 are sketched in figure 3.). Invoking the lemma (3.1), it is easy to get two similar inequalities corresponding to two different ranges of the parameter c_1, while c_2 is considered to be fixed and equal to U_{max}. For c_1 defined as an arbitrary real parameter from the interval $[U_{min}-U_\beta;\ U_{min}-U_\beta+(\frac{1}{2}U_\beta)^2(U_{max}-U_{min}+\frac{1}{2}U_\beta)^{-1}]$ and $c_\pm$ given by (2.18), the inequality takes the form

$$[c_r - \tfrac{1}{2}(c_1+U_{max})]^{\frac{1}{2}} + \left[1+\frac{k^2}{G}(c_+-U_{min})(U_{min}-c_-)\right]c_i^2 \leqslant [\tfrac{1}{2}(U_{max}-c_1)]^2. \tag{4.3a}$$

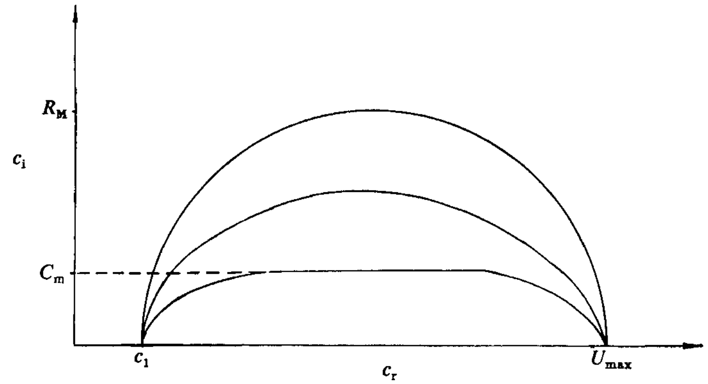

FIGURE 4. An illustration to the inequalities (4.3). A sketch of the semioval-like figures prescribed by (4.3) in comparison to the semiellipses and the semicircles (4.1) taken at a certain value of c_1 smaller than Miles' value.

Keeping the same value for c_2 and expressions for c_+, c_- we get the second inequality for $c_1 \in (-\infty : U_{min} - U_\beta]$:

$$[c_r - \tfrac{1}{2}(c_1 + U_{max})]^2 + \left[1 + \frac{k^2}{G}(c_+ - U_{max})(U_{max} - c_-)\right] c_i^2 \leqslant [\tfrac{1}{2}(U_{max} - c_1)]^2. \quad (4.3b)$$

We note that (4.3*a*) and (4.3*b*) coincide at the mutual point, i.e. when $c_1 = U_{min} - U_\beta$.

Thorough analysis of the one-parametric families of the domains prescribed by (4.3*a*, *b*) goes beyond the scope of this paper.

We shall present a qualitative picture and study the limiting cases in more detail. First we note that the inequalities (4.3*a*, *b*) can be enforced and at the same time considerably simplified by substituting G_0 for G. Then we get a one-parametric family of semiellipses, which on the one hand allows a straightforward detailed analysis, and on the other hand gives an idea of the way the boundary curves of the unsimplified inequalities (4.3*a*, *b*) behave. Thus, we can conclude that (4.3*a*, *b*) prescribe a one-parametric family of semioval-like domains, which lie within the corresponding semiellipses, which in turn lie within the semicircles (4.1) (see figure 4).

Let us consider in more detail the limiting cases:

(*A*) $c_1 \to -\infty$,

(*B*) $c_1 \to U_{min} - \tfrac{1}{2}U_\beta + (\tfrac{1}{2}U_\beta)^2 (U_{max} - U_{min} + \tfrac{1}{2}U_\beta)^{-1}$.

Case A. When $c_1 \to -\infty$ it follows from (4.3*b*) that

$$(U_{max} - c_r)|c_1| \geqslant c_i^2 \frac{k^2}{G} \tfrac{1}{2} U_\beta |c_1|$$

and finally we get

$$c_i^2 \leqslant \frac{2G}{\beta}(U_{max} - c_r).$$

Thus, in this case we have obtained the inequality (3.15) derived earlier. Within this context, it should be interpreted as exhibiting the tendency of our semioval-like figures to elongate along the real axis as the parameter c_1 decreases. When c_1 tends to minus infinity, the width of the figures also tends to infinity and we get our quasi-parabola bound.

Case B. When c_1 lies near Miles' value

$$U_{\min}-\tfrac{1}{2}U_\beta+(\tfrac{1}{2}U_\beta)^2(U_{\max}-U_{\min}+\tfrac{1}{2}U_\beta)^{-1}$$

(we denote the difference as α) the following inequality can be obtained:

$$c_i^2 \leqslant R_M^2 \left\{ \frac{1+\dfrac{\alpha^2}{4r_M^2}+\dfrac{\alpha}{R_M}}{1+\alpha\dfrac{k^2}{G}(U_{\max}-U_{\min}+U_\beta)} \right\}_{\min}, \quad \alpha \in [0;U_\beta] \tag{4.4}$$

The value of α should be chosen to minimize $(c_i)_{\max}$. The bound (4.4) is meaningful when the minimal value of the terms in the braces in (4.4) appears to be smaller than unity. We note that the inequality remains valid after substitution of G_0 for G, and then the modified ratio in braces should be minimized.

4.3. c_2 *is varied*, c_- *is fixed*

In the inequalities (4.3*a*, *b*), we have fixed c_2 and put it equal to $U_{\max}$, varied c_1 and used the inequality (3.1). Here we shall make use of the inequality (2.7). We shall fix the value of c_-, and put it equal to $U_{\min}$, while c_2 will be varied from $U_{\max}$ to infinity. Then, similarly to (4.3), we obtain two inequalities:

$$\left.\begin{aligned} &c_1+c_2 \leqslant U_{\max}+U_{\min}, \\ &[c_r-\tfrac{1}{2}(c_1+c_2)]^2+c_i^2 \leqslant [\tfrac{1}{2}(c_2-c_1)]^2-(c_2-U_{\max})(U_{\max}-c_1)\gamma, \\ &c_1+c_2 \geqslant U_{\max}+U_{\min}, \end{aligned}\right\} \tag{4.5a}$$

$$[c_r-\tfrac{1}{2}(c_1+c_2)]^2+c_i^2 \leqslant [\tfrac{1}{2}(c_2-c_1)]^2-(c_2-U_{\min})(U_{\min}-c_1)\gamma, \tag{4.5b}$$

where
$$\gamma = \left(1+\frac{4k^2}{\pi^2}\right)^{-1}; \quad c_1 = U_{\min}-\tfrac{1}{2}U_\beta+(\tfrac{1}{2}U_\beta)^2(c_2-U_{\min}+\tfrac{1}{2}U_\beta)^{-1}. \tag{4.5c}$$

The inequalities (4.5*a*, *b*) prescribe the one-parametric families of the semicircle domains. The semicircles (4.5) differ from Miles' one (2.19) both by the centre position on the real axis and the radius value. The mutual domain of all these semicircles is the domain in question, which gives the new constraints on c that we are seeking.

Let us start with the analysis of the inequalities (4.5*a*). Firstly, we note that the condition $c_1+c_2 \leqslant U_{\max}+U_{\min}$ restricts from above a range of permitted γ:

$$\gamma \leqslant 1-\sigma; \quad \sigma = R_H/2R_M \tag{4.6a}$$

and therefore a range of k from below

$$k \geqslant \tfrac{1}{2}\pi\frac{\sigma}{1-\sigma}. \tag{4.6b}$$

We also note that all the semicircumferences (4.5*a*) intersect at the same point with the abscissa c_{ri},

$$c_{ri} = c_1+2gR_M \tag{4.7}$$

This implies that the domain in question D is equivalent to the intersection (overlapping) of the two semicircles, one of which is that of Miles (2.20), while the

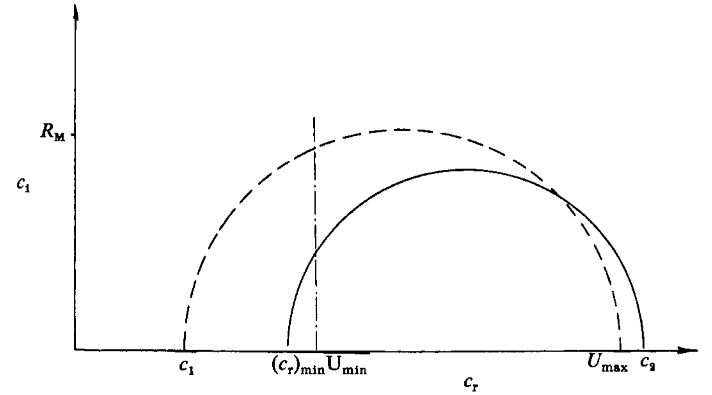

FIGURE 5. An illustration of the inequalities (4.5). The domain (4.10) for a certain fixed c_2 ($c_2 > U_{max}$) with condition (4.6). Miles' semicircle is shown by a dashed line.

second one will be specified below. It is convenient to present the results of this section using the notation

$$S(x, \rho)$$

for a circle with its centre at position x on the real axis and radius ρ. There are two ranges of k with qualitatively different D geometry:

(i) $k > \frac{1}{2}\pi \quad (\gamma < \frac{1}{2})$,

$$D = S(\tfrac{1}{2}(U_{max} + c_1), R_M) \cap S(\tfrac{1}{2}(U_{max} + U_{min}), \tfrac{1}{2}(U_{max} + U_{min}) - c_1). \tag{4.8}$$

It should be pointed out that the centre of the second semicircle is located at the same point as the Pedlosky one, but with a radius which is always greater than R_M but smaller than R_P for β large enough. The point of intersection lies to the left of the centre of Miles' semicircle.

(ii)
$$\tfrac{1}{2}\pi\frac{\sigma}{1-\sigma} \leqslant k \leqslant \tfrac{1}{2}\pi \quad (\tfrac{1}{2} \leqslant \gamma \leqslant 1-\sigma), \tag{4.9}$$

$$D = S(\tfrac{1}{2}(U_{max} + c_1), R_M) \cap S(c_1 + 2R_M\gamma, 2R_M[\gamma(1-\gamma)]^{\frac{1}{2}}). \tag{4.10}$$

In this range the second semicircle always has a radius value smaller than R_M. The point of intersection lies within the right-hand side of Miles' semicircle (see figure 5). Taking the latter fact into account we immediately find a new growth-rate bound:

$$c_i \leqslant 2R_M[\gamma(1-\gamma)]^{\frac{1}{2}}. \tag{4.11a}$$

It should be noted that γ can vary from $\frac{1}{2}$ to $\frac{3}{4}$. At the right-hand end of the interval (4.9) ($k \rightarrow \frac{1}{2}\pi$, $\gamma \rightarrow \frac{1}{2}$) the bound (4.11$a$) tends from below to Miles' value R_M and thus yields only a slight refinement of (2.20). Let us analyse the other end of the interval:

$$k \sim \tfrac{1}{2}\pi\frac{\sigma}{1-\sigma}; \quad \sigma = R_H/2R_M.$$

The minimum value of the square root in (4.11a) is attained at

$$\gamma = 1 - \sigma = \tfrac{3}{4}$$

and appears to be equal to $\frac{1}{4}\surd 3$.

Thus we have found that for a certain spatial scale specified by (4.9), the maximum growth rate: (i) increases slower with the increase of β than Miles' radius (2.21) does; (ii) appears to be bounded by a smaller limit value

$$c_i \leqslant \sqrt{3}\, R_H = \tfrac{1}{2}\sqrt{3}\, (U_{max} - U_{min}). \tag{4.11b}$$

Let us turn now to the inequalities (4.5*b*). It is easy to show that though they improve Miles' semicircle bound (2.20), they have no advantages over the intersection of (2.12), (2.20) and therefore will not be analysed here.

Thus the inequalities (4.8), (4.10), (4.11*a*) that we have derived here noticeably improve Miles' theorem (2.20). It seems that the most interesting result is the simple formula (4.11*a*) for the growth-rate bound.

To clarify the nature of these results, which were found without using any new 'basic' inequalities besides those of Miles (2.16) and Pedlosky (2.7), we recall that (4.5*a*, *b*) are a *barotropic–baroclinic generalization* of Miles' theorem. It is easy to see that in the purely baroclinic problem, i.e. when the characteristic horizontal scale L tends to infinity, then $\gamma \rightarrow 0$, and we come back to (2.20) with no improvement.

5. On the relations between the semicircle and the growth-rate-type evaluations

In this section we shall try to establish some relations between the different types of bounds of c from the previous sections, and to derive some new ones.

Let us revise the derivation of Pedlosky's semicircle (2.12). We recall that the last term in (2.11) has been evaluated by using the inequality (2.10). Here we represent this term in the form of a sum as follows:

$$\langle j[U - \tfrac{1}{2}(U_{max} + U_{min})]\rangle = \langle j(U - \alpha U_{min})\rangle - \langle j\{\tfrac{1}{2}[(1-2\alpha)\, U_{min} + U_{max}]\}\rangle, \tag{5.1}$$

where $\alpha \in (-\infty\,;1]$ is a certain dimensionless parameter. Making use of the positivity of $U - \alpha U_{min}$ and the lemma (3.1) we get the inequality

$$\langle j(U - \alpha U_{min})\rangle \geqslant (1-\alpha)\, U_{min} \frac{c_i^2}{G} \langle P\rangle. \tag{5.2}$$

Substituting (5.1) into (2.11), taking (2.7), (5.2) into account, we obtain

$$[c_r - \tfrac{1}{2}(U_{max} + U_{min})]^2 + \left[1 + \frac{\beta(1-\alpha)\, U_{min}}{G}\right] c_i^2 \leqslant [\tfrac{1}{2}(U_{max} - U_{min})]^2 + \frac{\beta}{k^2 + \frac{1}{4}\pi^2}\{\tfrac{1}{2}[U_{max} + U_{min}(1-2\alpha)]\}. \tag{5.3}$$

This inequality yields a one-parametric family of semiellipse-like figures. In spite of the fact that the domain confined by these figures can be easily analysed, we are interested here only in the analysis of the consequence of (5.3), which is obtained from (5.3) via replacing G by G_0:

$$[c_r - \tfrac{1}{2}(U_{max} + U_{min})]^2 + \left[1 + \frac{\beta(1-\alpha)\, U_{min}}{G_0}\right] c_i^2 \leqslant [\tfrac{1}{2}(U_{max} - U_{min})]^2 + \frac{\beta}{k^2 + \frac{1}{4}\pi^2}(\tfrac{1}{2}[U_{max} + U_{min}(1-2\alpha)]). \tag{5.4}$$

The inequality (5.4) prescribes a domain of allowed eigenvalues c lying within the inner boundary of the parametric family of semiellipses (5.4). As the parameter α

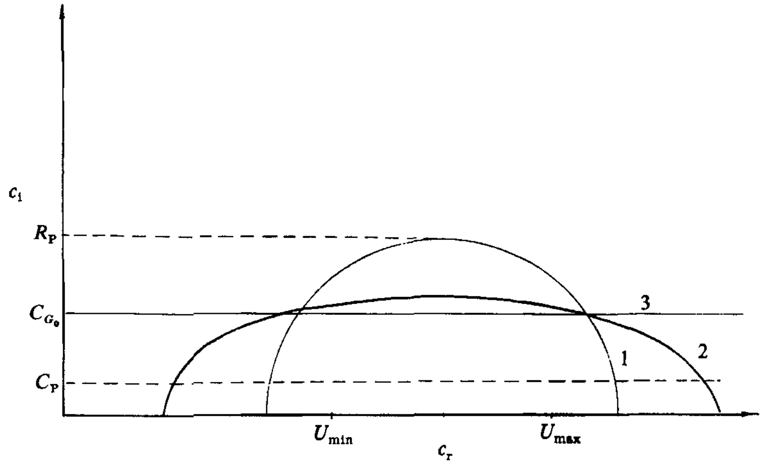

FIGURE 6. A sketch illustrating the tendency of the semiellipses (5.4) to elongate along the c_r axis as the parameter α decreases, while preserving the same intersection points. The semiellipse 1 (corresponding to $\alpha = 1$) coincides with the semicircle (2.12). The semiellipse 2 is taken for an intermediate value of $\alpha (\alpha < 1)$. The straight line 3 is the result of the semiellipse degeneration as $\alpha \to -\infty$. Pedlosky's straight line (2.15) (depicted by the lower dashed line) lies below the $\frac{1}{2}R_H$ value under condition (5.6c).

decreases, the semiellipses elongate horizontally (along the c_r axis) and become more oblate in the vertical direction: from the semicircle (2.12) when $\alpha = 1$, to degeneration into a straight line when $\alpha \to -\infty$ (see figure 6). The peculiar feature of these semiellipses is that they all share the same intersection point with the ordinate $c_i = C_G (C_G^2 = G_0/(k^2 + \frac{1}{4}\pi^2))$ under the condition

$$R_P^2 = R_H^2 + \beta R_H/(k^2 + \tfrac{1}{4}\pi^2) \geqslant C_G^2. \tag{5.5}$$

Let us consider three cases when the condition (5.5) is definitely satisfied

(a) Assume β to be large:

$$\beta \geqslant 8G_0/(U_{max} - U_{min}); \tag{5.6a}$$

then (5.5) holds true for arbitrary k.

(b) If we confine ourselves to consideration of small-scale disturbances only

$$k^2 \geqslant 4q/(U_{max} - U_{min}) - \tfrac{1}{4}\pi^2, \tag{5.6b}$$

then (5.5) holds true for arbitrary but non-zero β.

(c) Consider the flows to be smooth:

$$G_0 \leqslant \tfrac{1}{4}\pi^2[\tfrac{1}{2}(U_{max} - U_{min})]^2 \tag{5.6c}$$

then (5.5) is fulfilled for both k and β arbitrary.

Let us assume (5.5) to be fulfilled through one of the sufficient conditions (5.6a–c). Thus the domain of instability prescribed by (5.4) appears to be founded by the arcs of Pedlosky's semicircle from the left and right and by the limiting straight line from above. It is easy to see that the inequality (5.4) gives no advantage compared to the combination of Pedlosky's bounds, as the bound (2.15) obviously lies below the line $c_i = C_G$. But the fact that on the one hand, the line $c_i = C_G$ crosses the semicircle (2.12) according to (5.5), and on the other hand, C_G is $(4G_0/q)^{\frac{1}{2}}$ times greater than C_P of the bound (2.15), allows us to relate these independent Pedlosky's bounds:

$$R_P \geqslant MC_P \quad [M^2 = 4G_0/q = 2(1 + l/q^{\frac{1}{2}})] \tag{5.7}$$

This relation between Pedlosky's two bounds seems to be of some intrinsic interest, as it demonstrates the presence of additional constraints imposed by inner connections among flow parameters. Let us consider the consequences of the established relation in more detail. The inequality (5.7) literally means that Pedlosky's bound (2.15) cuts off a considerable portion of the semicircle (2.12) under condition (5.5). According to (5.6*a*, *b*) this occurs, for large β, for the whole spectrum of disturbances, and, for arbitrary β, for small-scale disturbances.

For the case of smooth flows specified by (5.6*c*) we get from (5.7), (5.6*c*)

$$c_i \leqslant \frac{R_H}{M} = \frac{U_{max} - U_{min}}{2(2(1+l/q^{\frac{1}{2}}))^{\frac{1}{2}}}. \quad (5.8)$$

It can also be easily shown by virtue of the composite quasi-parabola bound (3.16), that the upper bound $(c_i)_{max}$ for c_i is attained at $\beta \rightarrow 0$ and decreases monotonically from $(U_{max} - U\text{min})/2M$ down to zero, as β increases from zero to infinity.

We mention that for a class of smooth flows satisfying a less restrictive condition than (5.6*c*), namely

$$q \leqslant \tfrac{1}{4}\pi^2(U_{max} - U_{min})^2, \quad (5.9)$$

we can easily obtain an upper bound for c_i, which also decreases monotonically with β from $\frac{1}{2}(U_{max} - U_{min})$ at $\beta \rightarrow 0$ down to zero as $\beta \rightarrow \infty$.

6. Conclusions and discussion

6.1. *Summary of results*

The main results of the paper can be summarized as follows:

(i) New 'quasi-parabola' bounds on the complex phase speed c of the unstable modes have been found (inequalities (3.15), (3.16), figure 2), which effectively confine the domain of allowed c for a wide range of β.† The allowed domain diminishes as β increases and, it should be emphasized, tends to zero as β tends to infinity.

(ii) Miles' semicircle theorem, proved originally for Charney's model, has been extended for the general case of barotropic–baroclinic instability (inequality (2.20)) and modified (inequalities (4.3*a*, *b*), (4.4), (4.5*a*, *b*), (4.8), (4.10), (4.11*a*)). This results in a diminishing of the instability region along both the c_r (from the lower-value side) and c_i axes.

(iii) Some new relations between Pedlosky's two types of bound have been revealed, which makes it possible to impose new restrictions on c_i (inequality (5.8)) for the case of a sufficiently smooth flow specified by the inequalities (5.6*c*), (5.9).

This set of results taken as a whole allows us to remove some weak points of the previously known evaluations of the complex phase speed c and provides considerably sharper constraints on c.

6.2. *Discussion of results*

In addition to the rigorous bounds on c derived in the paper and listed above, the same integral relations can also provide some conclusions of a qualitative character, which we believe merit mentioning and brief discussion.

† We note that the lemma (3.1) and the extension of Miles' theorem (2.20), which, for the sake of simplicity, have been proved here for the case of a flat bottom ($\partial_y \eta = 0$), also hold for the presence of topography. Thus all the consequences of (3.1) and (2.20) can be extended in a straightforward manner to that case as well.

6.2.1. *East–West flow asymmetry*

First we want to clarify the question of how the east-west flow asymmetry exhibits itself in instability parameters. We also want to shed some light on the closely related question regarding the role of β, as in some works (see Killworth 1980 for a review of this issue and references) one encounters inferences that β destabilizes westward flows, a conclusion was based on the direct numerical analysis of the boundary-value problem for some particular examples.

It is obvious from our basic equations (2.5), (2.6), which are valid for both eastward and westward flows, that the sign of the flow velocity is only important for the instability parameters in the terms containing β explicitly. Eastward/westward flow asymmetry manifests itself most clearly in the shift of the instability domain on the c-plane along the c_r axis. It is easy to see directly from (2.5) that the real parts of the disturbance phase velocities for westward flow $(c_r)_W$ are generally greater in modulo than their counterparts with the same wavenumber for an eastward flow with the same profile, $(c_r)_E$. This asymmetry certainly reveals itself in all our evaluations of c_r given above.

It should be pointed out that all our bounds on c_i, as well as those of other authors are symmetric, in contrast to the evident asymmetry of the boundary-value problem and to the results of numerical experiments cited by Killworth (1980). To understand this less trivial question, let us present the exact explicit expression for c_i, which can be obtained by inserting c_r from (2.8) into (2.6):

$$c_i^2 = \frac{\langle U^2P\rangle}{\langle P\rangle^2} - \frac{\langle UP\rangle^2}{\langle P\rangle^2} - \tfrac{1}{4}\beta^2\frac{\langle j\rangle^2}{\langle P\rangle^2} + \beta\frac{\langle j\rangle\langle UP\rangle}{\langle P\rangle^2} - \beta\frac{\langle jU\rangle}{\langle P\rangle}. \tag{6.1}$$

One can immediately conclude that β *stabilizes the flow independently of flow direction for large enough* β. When β is small the situation requires a more detailed analysis. Let us consider the two last terms in (6.1), which are linear in β and thus are the principal β-containing terms in this range. Let us define two 'effective velocities' $U_{\text{eff}(1)}$ and $U_{\text{eff}(2)}$ as follows:

$$U_{\text{eff}(1)} = \langle UP\rangle/\langle P\rangle; \quad U_{\text{eff}(2)} = \langle Uj\rangle/\langle j\rangle. \tag{6.2}$$

Then the terms of interest take the form

$$\beta\frac{\langle j\rangle}{\langle P\rangle}U_{\text{eff}(1)} - \beta\frac{\langle j\rangle}{\langle P\rangle}U_{\text{eff}(2)} = \beta\frac{\langle j\rangle}{\langle P\rangle}\{U_{\text{eff}(1)} - U_{\text{eff}(2)}\}, \tag{6.3}$$

which demonstrates that contributions due to β in c_i are in a rather subtle balance in the small-β range. *Thus the influence of small* β *can stabilize or destabilize both westward and eastward flow* depending on the flow profile.

Let us try to evaluate from above the total contribution of the terms in the braces in (6.3). For eastward flow, i.e. $U > 0$, evaluating the first term from above and the second one from below, we obtain

$$\beta\frac{\langle j\rangle}{\langle P\rangle}(U_{\max} - U_{\min}).$$

Similarly, for westward flow ($U < 0$)

$$-\beta\frac{\langle j\rangle}{\langle P\rangle}(-|U_{\min}| + |U_{\max}|).$$

	L (km)	U (m s^{-1})	S	β	β_1
Open ocean	10^3	$(1–5)\times 10^{-2}$	10^{-2}	2×10^{-3}–40	10^2
Strong oceanic currents	$(0.5–1)\times 10^2$	1	4–1	10^{-1}	1
Synoptic-scale oceanic currents (eddies)	10^2	10^{-1}	1	0.2	2
Atmospheric flows	5×10^3	20	4×10^{-2}	10^2	10^2

TABLE 1. Orders of magnitude of the basic parameters in typical situations. The following values of the 'hidden' fixed dimensional parameters (subscripts o and a refer to the oceanic or atmospheric flows, respectively) were taken. Buoyancy frequency n: $n_a = 10^{-2}$, $n_o = 2\times 10^{-3}$. Vertical scale D: $D_a = 10$ km, $D_o = 5$ km; $f_o = 10^{-4}$ s^{-1}. Dimensional β: $\beta_o = 10^{-11}$ m^{-1} s^{-1}.

This analysis has been given in detail to demonstrate that the symmetry in our bounds on c_i appears owing to the crudeness of the existing estimations of $U_{\text{eff}(t)}$, and thus to identify the key point for further analyses within the same paradigm.

6.2.2. *Relevance to real geophysical flows*

Discussion of the role of our results in the prediction of real geophysical phenomena lies far beyond the scope of this paper. It should be stressed that we have confined ourselves to investigation of *a priori* bounds on eigenvalues to a rather idealized linear inviscid model, which, nevertheless, forms the basis for the majority of studies of geophysical flow dynamics. We note that, even within the linear theory framework, viscosity can qualitatively affect the instability parameters. The method based on integral inequalities can be applied to the viscid problem as well, and this is the subject of another paper (Gnevyshev & Shrira 1990). Nonlinearity is the factor of the greatest uncertainty. On the one hand we now have enough examples of when the linear theory is misleading from the very beginning (e.g. Romanova 1987), but on the other hand, there are many numerical experiments claiming the validity of the linear theory (see references, e.g. in Pedlosky 1979; Killworth 1980) for some particular examples. We shall not touch upon this question here and only mention that a comparison of our results with the data from numerical or *in situ* experiments can be useful, and the discrepancy between results can be informative, in particular to identify the manifestations of nonlinear mechanisms.

To facilitate the comparison of our evaluations with experimental data, it is convenient to locate the range relevant to different typical geophysical situations of the non-dimensional parameters β and S, which control the shape of the allowed domain of c. We recall that β and S were dimensionalized in §2.1 using typical scales of horizontal (L) and vertical (H) flow variability, and characteristic flow velocity (U). In terms of non-dimensional variables we distinguish two characteristic values of β, namely β_1 and β_{MP}, where β_1 is the threshold value defined by (3.18*a*) for the quasi-parabola bound (3.15)

$$\beta_1 \sim G_0/(U_{\max}-U_{\min}),$$

andβ^0_{MP} is the threshold value defined by (2.22)

$$\beta^0_{\text{MP}} = \pi_2(U_{\max}-U_{\min}) \approx 10^1.$$

To relate dimensional and non-dimensional parameters relevant to different types of geophysical flows we present table 1, which gives an idea of the orders of magnitude of some basic dimensional (U, L) and non-dimensional (β, S, β_1) parameters. It is easy to see from the table that ranges of both small ($\ll 1$) and large ($\gg 1$) values of non-dimensional β are relevant to geophysical flows.

When β is small (strong and narrow jets) all the evaluations an upper bound of the order of unity give for non-dimensional c_i (except the bound (2.15) for small k). Both Miles' and Pedlosky's semicircles lie close to that of Howard.

In spite of the fact that our results at $\beta \ll 1$ allow one to improve noticeably the known bounds, it seems not to be of special interest to dwell upon these improvements here, as in this range these new bounds are still far from being good in view of direct geophysical applications.

The situation is different for $\beta \gg 1$. We recall that in this range the different types of crude estimations of $(c_i)_{max}$ give roughly the following: Pedlosky's semicircle yields $O(\beta^{\frac{1}{2}})$, Miles' one gives $O(1)$, while the implications (3.22*a*), (3.24) of the quasi-parabola bound (3.15) provides $O(\beta^{-\frac{1}{2}})$ or even smaller values. This means that for typical atmospheric flows and open ocean currents our results yield at least a one-order gain in accuracy.

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